An introduction to Nichols algebras

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内容提要

1 Preliminaries

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Conventions

- $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $k < \theta \in \mathbb{N}_0$, then $\mathbb{I}_{k,\theta} = \{n \in \mathbb{N}_0 : k \leq n \leq \theta\}$
- $\mathbb{G}$ denotes the group of N-roots of unity in $\mathbb{k}$.
- $V^* := hom_{\mathbb{k}}(V, \mathbb{k})$,
- The finite field with q elements is denoted $\mathbb{F}_q$.

Groups

Theorem (Maschke)

Let G be a finite group. Then TFAE:

1. The characteristic of $\Bbbk$ does not divide $|G|$.
2. Every finite-dimensional representational of G completely reducible.

The tensor algebra

- Graded vector space with a fixed grading $V = \bigoplus_{n \in \mathbb{N}_0} V^n$
- Hilbert-Poincare series:

$$H_V = \sum_{n \in \mathbb{N}_0} \dim V^n t^n \in \mathbb{Z}[[t]] \qquad (1.1)$$

- The graded dual of a locally finite graded vector space $V = \bigoplus_{n \in \mathbb{N}_0} V^n$ is

$$V^* = \bigoplus_{n \in \mathbb{N}_0} V^{*n}, \quad V^{*n} = hom_{\mathbb{k}}(V^n, \mathbb{k}). \qquad (1.2)$$

The tensor algebra

- The product $\mu : T(V) \otimes T(V) \to T(V)$ of the tensor algebra is given by

$$\mu_{m,n} : T^m(V) \otimes T^n(V) \cong T^{m+n}(V) \tag{1.3}$$

- The enveloping algebra of a Lie algebra L:

$$U(L) := T(L)/ < xy - yx - [x, y] : x, y \in L > \tag{1.4}$$

- $\delta : V \to T(V) \otimes T(V), \delta(v) = v \otimes 1 + 1 \otimes v$ extends to $\Delta : T(V) \to T(V) \otimes T(V)$. Then $T(V)$ becomes a Hopf algebra.
- $\delta : L \to U(L) \otimes U(L), \delta(v) = v \otimes 1 + 1 \otimes v$ extends to $\Delta : U(L) \to U(L) \otimes U(L)$, so that $U(L)$ is a Hopf algebra.

The symmetric algebra

- $S(V) := T(V)/\langle xy - yx : x, y \in V\rangle = \bigoplus_{n>0} S^n(V)$
- $\Lambda(V) := T(V)/\langle xy + yx : x, y \in V\rangle = \bigoplus_{n>0} \Lambda^n(V)$

Coalgebras and Hopf algebras

- $D \wedge E := \{c \in C : \Delta(c) \in D \otimes C + C \otimes E\}$
- Simple coalgebra
- coradical
- cosemisimple coalgebra: $C = corad(C)$
- Pointed: $corad(C) = \bigoplus V_i$, where V_i are subcoalgebras of C with $\dim V_i = 1$

The tensor coalgebras

The tensor coalgebra $T^c(V)$ is the vector space $T(V)$ with Δ given by

$$\Delta(v_1 v_2 \dots v_n) = \sum_{j \in I_n} v_1 \dots v_j \otimes v_{j+1} \dots v_n \tag{1.5}$$

Gelfand-Kirillov dimension

GK-dimension

If A is finitely generated,

$$\mathrm{GKdim} A := \overline{\lim_{n\to\infty}} \log_n \dim A_n \tag{1.6}$$

If A is not finitely generated

$\mathrm{GKdim} A := \sup\{\mathrm{GKdim} B | B$finitely generated subalgebra of $A\}$

- $\mathrm{GKdim} T(V) = \infty$, if $\dim V > 1$
- $\mathrm{GKdim} S(V) = d$, if $\dim V = d$
- Let A be finitely generated,then $\mathrm{GKdim} A = 0 \Leftrightarrow \dim A < \infty$.
- $\mathrm{GKdim} U(L) = \dim L$, if L is a Lie algebra.

Braided vector spaces

Definition

We say (V, c) is a braided vector space if $c \in GL(V \otimes V)$ satisfies

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c) \tag{2.1}$$

Braided vector spaces

- symmetry: $c^2 = \mathrm{id}$
- Hecke type: $char\ \Bbbk = 0,\ k \in \Bbbk^\times, q \neq -1.$
  $(c - q\mathrm{id})(c + \mathrm{id}) = 0$
- Diagonal type: $c^q(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$
- Triangular type: $c(x_i \otimes x_j) \in q_{ij} x_j \otimes x_i + V_{j-1} \otimes V$
- Rack type

Braided vector spaces

Rack type

$(\Bbbk X, c^q)$ is of rack type

$$c^q(e_x \otimes e_y) = q_{x,y} e_{x \triangleright y} \otimes e_x \quad x, y \in X \tag{2.2}$$

such that

$$q_{x,y \triangleright z} q_{y,z} = q_{x \triangleright y, x \triangleright z} q_{x,z}, \quad x, y, z \in X \tag{2.3}$$

Let W be a vector space, $q : X \times X \to GL(W)$ be a function. $V = \Bbbk X \otimes W$, $e_x v := e_x \otimes v$,

$$c^q(e_x v \otimes e_y w) = e_{x \triangleright y} q_{x,y}(w) \otimes e_x v, \quad x, y \in X, v, w \in W. \tag{2.4}$$

Racks

Permutation rack

Let X be a non-empty set. Given $\sigma \in \mathbb{S}_X$, the associated permutation rack $(X, \triangleright)$ is defined by $x \triangleright y = \sigma(y)$

Group as a rack

A group G is a rack with $x \triangleright y = xyx^{-1}$. If $X \subset G$ is a conjugacy class, then X is a subrack of G.

Twisted conjugacy rack

Let G be a group and $T \in \operatorname{Aut}(G)$. Let $\rightharpoonup_T$ be the action of G on itself given by $x \rightharpoonup_T y = xyT(x^{-1}), x, y \in G$. Then the orbit $\mathcal{O}_x^{G,T}$ of $x \in G$ by this action is a rack with operation

$$y \triangleright_T z = yT(zy^{-1}), \quad y, z \in \mathcal{O}_x^{G,u} \tag{2.5}$$

The rack $(\mathcal{O}_x^{G,T})$ is called a twisted conjugacy rack.

Racks

Affine rack

Let A be an abelian group and $T \in \mathrm{Aut}(A)$. We define operation $\triangleright$ by

$$x \triangleright y = (1 - T)x + Ty, \quad x, y \in A. \tag{2.6}$$

Then (A, T) is a rack, denoted Aff(A, T).

Racks

Simple rack

A finite rack X is simple if

- it has at least 2 elements,
- for any surjective morphism of racks $\pi : X \to Y$, either π is an isomorphism or Y has just one element.

Racks

Classification of simple racks

Let X be a finite simple rack with $|X|$ elements. Then either of the following holds:

1. $|X|$ is divisible by at least 2 primes, In this case, there exist

- a simple non-abelian group L
- $t \in \mathbb{N}$, and
- $\theta \in \mathrm{Aut}L$

such that X is a twisted conjugacy class of type (G, T), where

- $G \in L^t$ and
- $T \in \mathrm{Aut}(L^t)$ acts by

$$T(\ell_1, \dots, \ell_t) = (\theta(\ell_t), \ell_1, \dots, \ell_{t-1}), \ell_1, \dots \ell_t \in L. \qquad (2.7)$$

Furthermore, L and t are unique, and T only depends on its conjugacy class in Out$(L^t) = \mathrm{Aut}(L^t)/Inn(L^t)$.

Racks

2. $|X| = p^t$ where p is prime and $t \in \mathbb{N}$. In this case, there exist 2 possibilities:

1. $t = 1$ and $X \cong \mathbb{I}_p$ is the permutation rack of the cycle $(1, 2, \dots, p)$
2. X is the affine rack $\mathbb{F}_p^t, T$, where T is the companion matrix of a monic irreducible polynomial $f \in \mathbb{F}_p[X]$ of degree t, different from X and $X - 1$.

Braided tensor categories

Braided monoidal category

A Braided monoidal category is a monoidal category $\mathcal{C}$ provided with a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$, called the braiding, that is required to fulfill the hexagon axioms.

Yetter-Drinfeld modules

Let H be a Hopf algebra with bijective antipode S. Let $G(H)$ be the group of group-like elements.

Yetter-Drinfeld modules

A Yetter-Drinfeld module over H is a vector space V provided with

1. a structure of left H-module $\mu : H \otimes V \to V$ and
2. a structure of left H-comodule $\rho : V \to H \otimes V$ such that
3. for all $h \in H$ and $v \in V$, the following compatibility condition holds:

$$\rho(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)} \tag{2.8}$$

Thus we have the category ${}^H_H\mathscr{Y}\mathscr{D}$ of Yetter-Drinfeld modules, with morphisms being linear maps that preserve both the action and coaction, i.e. both module maps and comodule maps.

Yetter-Drinfeld modules

Braiding of Yetter-Drinfeld modules

${}^H_H\mathscr{Y}\mathscr{D}$ is braided tensor category, with tensor product of modules and comodules and braiding

$$c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)} \quad V, W \in {}^H_H\mathscr{Y}\mathscr{D}, v \in V, w \in W.$$

Here $c_{V,W}$ is bijective because S is so; indeed

$$c^{-1}_{W,V}(v \otimes w) = w_{(0)} \otimes S^{-1}(w_{(-1)}) \cdot v, \quad V, W \in {}^H_H\mathscr{Y}\mathscr{D}, v \in V, w \in W.$$

Yetter-Drinfeld modules

Realization

Let (V, c) be a rigid braided space. Then there is a Hopf algebra $H(V)$ such that $V \in^{H(V)}_{H(V)} \mathcal{YD}$ and $c = c_{V,V}$.

Yetter-Drinfeld modules

YD-pair

YD-pairs classify the $V \in^H_H \mathcal{YD}$ with $\dim V = 1$.

$$\delta(k) = g \otimes k, \quad h \cdot k = \chi(h)k$$

principal realization

Let $q = (q_{ij}) \in (\mathbb{k})^{\mathbb{I}\times\mathbb{I}}$ be a 2-cocycle and let V be the corresponding braided vector space of diagonal type with respect to a basis $(x_i)_{i\in\mathbb{I}}$. A principal realization of (V, c) is a collection $(g_i, \chi_i)_{i\in\mathbb{I}}$ of YD-pairs such that $q_{ij} = \chi_j(g_i)$ for all $i, j \in \mathbb{I}$. But there might be realizations different from here.

Yetter-Drinfeld modules

Yetter-Drinfeld modules over finite abelian groups

$\Bbbk$ is algebaically closed and $\mathrm{Char}\Bbbk = 0$, If $H = \Bbbk\Gamma$, where Γ is finite abelian, then every $V \in {}^H_H\mathcal{YD}$ of dimension $\theta \in \mathbb{N}$ is determined by families $(\chi_i)_{i\in\mathbb{I}_\theta}$ and $(g_i)_{i\in\mathbb{I}_\theta}$

Yetter-Drinfeld modules

YD-triple(realization of a 2-block)

Let (g, χ, η) be a YD-triple. Let $\mathcal{V}_g(\chi, \eta)$ be a vector space with basis $(x_i)_{i \in \mathbb{I}_2}$, where the action and coaction are given by

$$h \cdot x_1 = \chi(h)x_1, \quad h \cdot x_2 = \chi(h)x_2 + \eta(h)x_1, \tag{2.9}$$

$$\delta(x_i) = g \otimes x_i, \qquad h \in H. \tag{2.10}$$

Then $\mathcal{V}_g(\chi, \eta) \in {}^H_H\mathcal{YD}$.

Yetter-Drinfeld modules

conjugacy class in a finite group

Let G be a finite group. Let $\mathcal{O}$ be a conjugacy class in G, pick $x \in \mathcal{O}$ and (W, ρ) an irreducible representation of $G^x = \{g \in G : gx = xg\}$. Let

$$M(\mathcal{O}, \rho) = \mathrm{Ind}_{G^x}^{G} \rho = \Bbbk G \otimes_{\Bbbk G^x} W \tag{2.11}$$

Then $M(\mathcal{O}, \rho) \in_{\Bbbk G}^{\Bbbk G} \mathcal{YD}$

Hopf algebras in braided tensor categories

- monoid(algebra) in a monoidal category $\mathcal{C}$
- comoid(coalgebra)in a monoidal category $\mathcal{C}$
- tensor product of 2 monoids in $\mathcal{C}$
- bialgebra in a monoidal category $\mathcal{C}$

Bosonization

Bosonization

H is a Hopf algebra and R is a braided Hopf algebra, then $R\#H$ is a Hopf algebra by

$$(r\#h)(s\#f) = r(h_{(1)}c \cdot s)\#h_{(2)}f, \tag{3.1}$$

$$\Delta(r\#h) = r^{(1)}\#(r^{(2)})_{(-1)}h_{(1)} \otimes (r^{(2)})_{(0)}\#h_{(2)} \tag{3.2}$$

We call $R\#H$ the bosonization (Radford biproduct)of R .

Nichols algebras: definitions

Nichols algebra

The Nichols algebra $\mathcal{B}(V)$ is the image of the map Ω.

Criterion using skew derivations

Let $x \in T^n(V), n \geq 2$. If $\partial_f(x) = 0$ for all $f \in V^*$, then $x \in J^n(V)$.

Nichols algebras: definitions

Application to the pointed Hopf algebras

Let A be a pointed Hopf algebra and let $gr\,A$ be the graded coalgebra associated to the coradical filtration. Then

$$gr\,A \cong \mathcal{R}\#\Bbbk G(A) \tag{3.3}$$

where $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}^n$ is a graded Hopf algebra in ${}^H_H\mathcal{YD}$. Set $V = \mathcal{R}^1$. $\mathcal{R}$ is a post-Nichols algebra of V, while its subalgebra generated by V is isomorphic to $\mathcal{B}(V)$.

Nichols algebras: definitions

Problems

- when is $\dim \mathcal{B}(V) < \infty$? For such V, classify its finite-dimensional post-Nichols algebras.
- When $\mathrm{GKdim}(\mathcal{B}(V)) < \infty$?For such V, classify its finite GK-dimensional post-Nichols algebras.

Conjecture

Assume $\mathrm{Char}\Bbbk = 0$ and H is semisimple. Let $V \in^H_H \mathcal{YD}$ such that $\dim \mathcal{B}(V) < \infty$. Then there is no finite-dimensional post-Nichols algebras except $\mathcal{B}(V)$ itself.

Nichols algebras: techniques

- direct computation
- dual
- twisting
- discard
- decomposition

Symmetries and Hecke type

Proposition 7

Let (V, c) be a braided vector space such that c is either a symmetry or of Hecke type with label $q \notin \mathbb{G}_\infty$. Then $\mathcal{B}(V) \cong T(V)/\langle \ker(c + \mathrm{id}) \rangle$

Diagonal type

Theorem 5

Let V be a braided vector space of Cartan type with Cartan matrix A. Then $\dim \mathcal{B}(V) < \infty \Leftrightarrow A$ is a finite Cartan matrix.

Conjecture

If $\mathrm{GKdim}\mathcal{B}(V) < \infty$, then its Weyl groupoid is finite.

Theorem 6

If either its Weyl groupoid is infinite and $\dim V = 2$, or else is of affine Cartan type, then $\mathrm{GKdim}\mathcal{B}(V) = \infty$.

Triangular type

Theorem 7

$\mathrm{GKdim}\mathcal{B}(V(\epsilon, \ell)) < \infty$ if and only if $\ell = 2$ and $\epsilon \in \{\pm 1\}$. If this happens, then $\mathrm{GKdim}\mathcal{B}(V(\epsilon, \ell)) = 2$

Theorem 8

Let V be a braided vector space with braiding (83). Then $\mathrm{GKdim}\mathcal{B}(V) < \infty$ if and only if the ghost is discrete and V is as in Table1.

Theorem 9

Let V be a braided vector space with braiding (84). Then $\mathrm{GKdim}\mathcal{B}(V) < \infty$ if and only if $\epsilon = -1$ and either of the following holds:

1. $q_{12}q_{21} = 1$ and $q_{22} = \pm 1$; in this case $\mathrm{GKdim}\mathcal{B}(V) = 1$
2. $q_{22} = -1 = q_{12}q_{21}$;in this case $\mathrm{GKdim}\mathcal{B}(V) = 2$.

Triangular type

Let G be an abelian group and $V \in^H_H \mathcal{YD}$ of dimension 3 but not of diagonal type. Then $\mathrm{GKdim}\mathcal{B}(V) < \infty$ if and only if V has the shape (83)(84).

Rack type, infinite dimension

- collapse
- type C, D, F
- kthulhu

Theorem 10

A rack X of type C, D or F collapses.

Question

Are the criteria of types C, D, F valid to finite GK-dimension?

Rack type, infinite dimension

- $\mathcal{O}_\sigma^{S_m}$,$\mathcal{O}_\sigma^{A_m}$
- unipotent(semisimple) conjugacy class in a Chevalley or Steinberg group.
- sporadic simple group different from the Moster M

Question

Are there cocycles for $SP_{2n,q}$ or $SU_{m,q}$ such that the corresponding Nichols algebras are finite dimensional?

Rack type, finite dimension

Thank you!

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